Elementarily Computable Functions Over the Real Numbers and R-Sub-Recursive Functions

Olivier Bournez, Emmanuel Hainry

Elementarily Computable Functions Over the Real Numbers and $\mathbb{R}$-Sub-Recursive Functions

Olivier Bournez and Emmanuel Hainry

LORIA/INRIA, 615 Rue du Jardin Botanique, BP101
54602 Villers lès Nancy, France
{Olivier.Bournez,Emmanuel.Hainry}@loria.fr

Abstract We present an *analog* and *machine-independent* algebraic characterization of elementarily computable functions over the real numbers in the sense of recursive analysis: we prove that they correspond to the smallest class of functions that contains some basic functions, and closed by composition, linear integration, and a simple limit schema.
We generalize this result to all higher levels of the Grzegorczyk Hierarchy.
This paper improves several previous partial characterizations and has a dual interest:

- Concerning *recursive analysis*, our results provide machine-independent characterizations of natural classes of computable functions over the real numbers, allowing to define these classes without usual considerations on higher-order (type 2) Turing machines.
- Concerning *analog models*, our results provide a characterization of the power of a natural class of analog models over the real numbers and provide new insights for understanding the relations between several analog computational models.

1 Introduction

Several approaches have been proposed to model computations over real numbers. *Recursive analysis* or *computable analysis*, was introduced by Turing [33], Grzegorczyk [15], Lacombe [18]. Many works have been devoted to giving computable foundations to most of the concepts of mathematical analysis in this framework : see *e.g.* monograph [34].

Alternative views exist. Among them, we can mention the model proposed by Blum *et al.*, sometimes called *real Turing machine*, measuring the algebraic complexity of problems independently of real number representation considerations defined in [5] and extended to arbitrary structures in [26]. Several papers have been devoted to understanding complexity classes and their relations in this framework: see monographs [4,26].

These models concern *discrete time* computability. Models of machines where the time is *continuous* can also be considered. The first ever built computers were continuous time machines: *e.g. Blaise Pascal's pascaline* or Lord Kelvin's model of *Differential Analyzer* [32], that gave birth to a real machine, built in 1931 at the MIT to solve differential equations [8], and which motivated Shannon's *General Purpose Analog Computer (GPAC) model* [30], whose computational power was characterized algebraically in terms of solutions of polynomial differential equations [30,27,19,14]. Continuous time machines also include analog neural networks [25,31], hybrid systems [3,6], or theoretical physical models [24,17,13]: see also survey [25].

The relations between all the models are not fully understood. One can say, that the theory of analog computations has not yet experienced the unification that digital discrete time computations have experienced through Turing work and the so-called *Church thesis* [11,25].

This however becomes a crucial matter since the progress of electronics makes the construction of some of the machines realistic, whereas some models were recently proved very (far too?) powerful: using the so-called *Zeno's paradox*, some models make it possible to compute non-Turing computable functions in a constant time: see *e.g.* [20,7,3,17,13].

Notice that understanding whether there exist analog continuous time models that do not suffer from Zeno's paradox problems is also closely related to the important problems of finding criteria for so-called *robustness* for continuous (hybrid) time models: see *e.g.* [16,2].

In [20], Moore introduced a class of functions over the reals inspired from the classical characterization of computable functions over integers: observing that the continuous analog of a primitive recursion is a differential equation, Moore proposes to consider the class of $\mathbb{R}$-*recursive functions*, defined as the the smallest class of functions containing some basic functions, and closed by composition, differential equation solving (called *integration*), and minimization.

This class of functions, also investigated in [21,22], can be related to GPAC computable functions: see [20], corrected by [14].

Putting aside possible objections about the physical feasibility of the μ-operator considered in paper [20], the original definitions of this class in [20] suffer from several technical problems[1]. At least some of them make it possible to use a *"compression trick"* (another incarnation of Zeno's paradox) to simulate in a bounded time an unbounded number of discrete transitions in order to recognize arithmetical reals [20].

In his PhD dissertation, Campagnolo [11] proposes to restrict to the (better-defined) subclass $\mathcal{L}$ of $\mathbb{R}$-recursive functions corresponding to the smallest class of functions containing some basic functions and closed by composition and *linear* integration. Class $\mathcal{L}$ is related to functions elementarily computable over integers in classical recursion theory and functions elementarily computable over the real numbers in recursive analysis (discussed in [35]): any function of class $\mathcal{L}$ is elementarily computable in the sense of recursive analysis, and conversely, any function over the integers computable in the sense of classical recursion theory is the restriction to integers of a function that belongs to $\mathcal{L}$ [11,10].

[1] For example not well defined functions are considered, $\infty \times 0$ is always considered as 0, etc.... Some of them are discussed in [11,10] and even in the original paper [20].

However, the previous results do not provide a characterization of *all* functions over the reals that are computable in the sense of recursive analysis.

This paper provides one:

Theorem 1. *For functions over the reals of class $\mathcal{C}^2$ defined on a product of compact intervals with rational endpoints, f is elementarily computable in the sense of recursive analysis iff it belongs to the smallest class of functions containing some basic functions and closed by composition, linear integration and a simple limit schema.*

We extend this theorem to a characterization of all higher levels of the Grzegorczyk hierarchy.

Theorem 2. *For functions over the reals of class $\mathcal{C}^2$ defined on a product of compact intervals with rational endpoints, f is computable in the sense of recursive analysis in level $n \geq 3$ of the Grzegorczyk hierarchy iff f belongs to the smallest class of functions containing some (other) basic functions and closed by composition, linear integration and a simple limit schema.*

Concerning *analog models*, these results have several impacts: first, they contribute to understand analog models, in particular the relations between GPAC computable functions, $\mathbb{R}$-recursive functions, and computable functions in the sense of recursive analysis. Furthermore, they prove that no *Super-Turing* phenomenon can occur for these classes of functions. In particular we have a "robust" class of functions in the sense of [16,2].

Concerning *recursive analysis*, our theorems provide *a purely algebraic* and *machine independent* characterization of elementarily computable functions over the reals. Observe the potential benefits offered by these characterizations compared to classical definitions of these classes in recursive analysis, involving discussions about higher-order (type 2) Turing machines (see *e.g.* [34]), or compared to characterizations in the spirit of [9].

In Section 2, we start by some mathematical preliminaries. In Section 3, we recall some notions from classical recursion theory. We present basic definitions of recursive analysis in Section 4. Previous known results are recalled in Section 5. Our characterizations are presented in Section 6. The proofs are given in Sections 7 and 8. Some extensions are presented in Section 9.

2 Mathematical preliminaries

Let $\mathbb{N}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{R}^{>0}$ denote the set of natural integers, the set of rational numbers, the set of real numbers, and the set of positive real numbers respectively. Given $x \in \mathbb{R}^n$, we write $\overrightarrow{x}$ to emphasize that x is a vector.

We will use the following simple mathematical result

Lemma 1. *Let $F : \mathbb{R} \times \mathcal{V} \subset \mathbb{R}^{k+1} \to \mathbb{R}^l$ be a function of class*[2] *$\mathcal{C}^1$, and $\beta(x) : \mathcal{V} \to \mathbb{R}$ be some continuous function. Assume that for all t and $\overrightarrow{x} = (x_1, \dots, x_k)$, $\frac{\partial^2 F}{\partial t \partial x_i}(t, \overrightarrow{x})$ exists and*

- *$\|\frac{\partial F}{\partial t}(t, \overrightarrow{x})\| \leq K \exp(-t\beta(\overrightarrow{x}))$,*
- *and $\|\frac{\partial^2 F}{\partial t \partial x_i}(t, \overrightarrow{x})\| \leq K \exp(-t\beta(\overrightarrow{x}))$ for some constant $K > 0$.*

Let $\mathcal{D}$ be the subset of the $\overrightarrow{x} \in \mathcal{V}$ with $\beta(\overrightarrow{x}) > 0$.

Then:

- *For all $\overrightarrow{x} \in \mathcal{D}$, $F(t, \overrightarrow{x})$ has a limit $L(\overrightarrow{x})$ in $t = +\infty$.*
- *Function $L(\overrightarrow{x})$ is of class $\mathcal{C}^1$.*
- *Its partial derivative $\frac{\partial L}{\partial x_i}$ are the limit of $\frac{\partial F}{\partial x_i}(t, \overrightarrow{x})$ in $t = +\infty$.*
- *Furthermore*
$$\|F(t, \overrightarrow{x}) - L(\overrightarrow{x})\| \leq \frac{K \exp(-t\beta(\overrightarrow{x}))}{\beta(\overrightarrow{x})}$$
and
$$\|\frac{\partial F}{\partial x_i}(t, \overrightarrow{x}) - \frac{\partial L}{\partial x_i}(\overrightarrow{x})\| \leq \frac{K \exp(-t\beta(\overrightarrow{x}))}{\beta(\overrightarrow{x})}.$$

Proof. By mean value theorem,

$$\begin{aligned} \|F(t, \overrightarrow{x}) - F(t', \overrightarrow{x})\| &\leq \int_t^{t'} K \exp(-t\beta(\overrightarrow{x})) \\ &\leq K \int_t^{+\infty} \exp(-t\beta(\overrightarrow{x})) = K \tfrac{\exp(-t\beta(\overrightarrow{x}))}{\beta(\overrightarrow{x})}. \end{aligned}$$

This implies that $F(t, \overrightarrow{x})$ satisfies Cauchy criterion, and hence converges in $t = +\infty$. The first inequality of last item is obtained by letting t' go to $+\infty$. Observe that it implies that the convergence is uniform in $\overrightarrow{x}$ in every compact on which $\beta(\overrightarrow{x}) \geq \epsilon$ for some ϵ, in particular given $\overrightarrow{x}$ on the compact $C_{\overrightarrow{x}}$ defined by the set of $\overrightarrow{x}'$ with $\beta(\overrightarrow{x}') \geq \beta(\overrightarrow{x})/2$.

Replacing $F(t, \overrightarrow{x})$ by $\frac{\partial F}{\partial x_i}(t, \overrightarrow{x})$ in previous arguments proves the uniform convergence of $\frac{\partial F}{\partial x_i}(t, \overrightarrow{x})$ in $t = +\infty$ on $C_{\overrightarrow{x}}$ for all i.

Observing that the derivative of a converging sequence of functions, whose sequence of derivatives converges uniformly, exists and is the limit of the derivatives, and that the limit of a uniformly converging sequence of continuous functions is continuous, the other assertions follows.

The following result[3], with previous lemma, is a key to provide upper bounds on the growth of functions of our classes (c.f. Lemma 7).

[2] Recall that function $f : \mathcal{D} \subset \mathbb{R}^k \to \mathbb{R}^l$, $k, l \in \mathbb{N}$, is said to be of class $\mathcal{C}^r$ if it is r-times continuously differentiable on $\mathcal{D}$. It is said to be of class $\mathcal{C}^\infty$ if it is of class $\mathcal{C}^r$ for all r.

[3] As it was already the case in Campagnolo's Dissertation.

Lemma 2 (Bounding Lemma for Linear Differential Equations [1]). *For linear differential equation $\overrightarrow{x}' = A(t)\overrightarrow{x}$, if A is defined and continuous on interval $I = [a, b]$, where $a \leq 0 \leq b$, then, for all $\overrightarrow{x}_0$, the solution of $\overrightarrow{x}' = A(t)\overrightarrow{x}$ with initial condition $\overrightarrow{x}(0) = \overrightarrow{x}_0$ is defined and unique on I. Furthermore, the solution satisfies*

$$\|\overrightarrow{x}(t)\| \leq \|\overrightarrow{x}_0\| \exp(\sup_{\tau\in[0,t]} \|A(\tau)\| t).$$

Remark 1. Recall that the solution of any differential equation of type $\overrightarrow{x}' = A(t)\overrightarrow{x} + B(t)$, $\overrightarrow{x}(0) = \overrightarrow{x}_0$, where $A(t)$ is a $n \times n$ matrix and $B(t)$ is a n dimension vector can be obtained by the solution of linear differential equation $\overrightarrow{y}' = C(t)\overrightarrow{y}$, $\overrightarrow{y}(0) = \overrightarrow{y}_0$ by working in dimension $n + 1$ and considering

$$y(t) = \begin{pmatrix} x(t) \\ 1 \end{pmatrix}, y_0 = \begin{pmatrix} x_0 \\ 1 \end{pmatrix}, \ \ \textit{and } C = \begin{pmatrix} A & B \\ 0 & 0 \end{pmatrix}.$$

3 Classical Recursion Theory

Classical recursion theory deals with functions over integers. Most classes of classical recursion theory can be characterized as closures of a set of basic functions by a finite number of basic rules to build new functions [29,23]: given a set $\mathcal{F}$ of functions and a set $\mathcal{O}$ of operators on functions (an operator is an operation that maps one or more functions to a new function), $[\mathcal{F}; \mathcal{O}]$ will denote the closure of $\mathcal{F}$ by $\mathcal{O}$.

Proposition 1 (Classical settings: see *e.g.* **[29,23]).** *Let f be a function from $\mathbb{N}^k$ to $\mathbb{N}$ for $k \in \mathbb{N}$. Function f is*

- *elementary iff it belongs to $\mathcal{E} = [0, S, U, +, \ominus; \mathrm{COMP}, \mathrm{BSUM}, \mathrm{BPROD}]$;*
- *in class $\mathcal{E}_n$ of the Grzegorczyk Hierarchy ($n \geq 3$) iff it belongs to $\mathcal{E}_n = [0, S, U, +, \ominus, E_{n-1}; \mathrm{COMP}, \mathrm{BSUM}, \mathrm{BPROD}]$;*
- *primitive recursive iff it belongs to $\mathcal{PR} = [0, U, S; \mathrm{COMP}, \mathrm{REC}]$;*
- *recursive iff it belongs to $\mathcal{R}ec = [0, U, S; \mathrm{COMP}, \mathrm{REC}, \mathrm{MU}]$.*

A function $f : \mathbb{N}^k \to \mathbb{N}^l$ is elementary (resp: primitive recursive, recursive) iff its projections are elementary (resp: primitive recursive, recursive).

The base functions $0, (U_i^m)_{i,m\in\mathbb{N}}, S, +, \ominus$ and the operators COMP, BSUM, BPROD, REC, MU *are given by*

1. $0 : \mathbb{N} \to \mathbb{N}$, $0 : n \mapsto 0$; $U_i^m : \mathbb{N}^m \to \mathbb{N}$, $U_i^m : (n_1, \dots, n_m) \mapsto n_i$; $S : \mathbb{N} \to \mathbb{N}, S : n \mapsto n + 1$; $+ : \mathbb{N}^2 \to \mathbb{N}$, $+ : (n_1, n_2) \mapsto n_1 + n_2$; $\ominus : \mathbb{N}^2 \to \mathbb{N}$, $\ominus : (n_1, n_2) \mapsto max(0, n_1 - n_2)$;
2. BSUM *: bounded sum. Given f, $h = \mathrm{BSUM}(f)$ is defined by $h : (\overrightarrow{x}, y) \mapsto \sum_{z<y} f(\overrightarrow{x}, z)$;* BPROD *: bounded product. Given f, $h = \mathrm{BPROD}(f)$ is defined by $h : (\overrightarrow{x}, y) \mapsto \prod_{z<y} f(\overrightarrow{x}, z)$;*
3. COMP *: composition. Given f and g, $h = \mathrm{COMP}(f, g)$ is defined as the function verifying $h(\overrightarrow{x}) = g(f(\overrightarrow{x}))$;*

4. REC *: primitive recursion . Given f and g, $h = \mathrm{REC}(f, g)$ is defined as the function verifying $h(\overrightarrow{x}, 0) = f(\overrightarrow{x})$ and $h(\overrightarrow{x}, n+1) = g(\overrightarrow{x}, n, h(\overrightarrow{x}, n))$.*
5. MU *: minimization. The minimization of f is $h : \overrightarrow{x} \mapsto \inf\{y : f(\overrightarrow{x}, y) = 0\}$.*

Functions E_n, involved in the definition of the classes $\mathcal{E}_n$ of the Grzegorczyk Hierarchy, are defined by induction as follows (when f is a function, $f^{[d]}$ denotes its d-th iterate: $f^{[0]}(\overrightarrow{x}) = x$, $f^{[d+1]}(\overrightarrow{x}) = f(f^{[d]}(\overrightarrow{x}))$):

1. $E_0(x, y) = x + y$,
2. $E_1(x, y) = (x+1) \times (y+1)$,
3. $E_2(x) = 2^x$,
4. $E_{n+1}(x) = E_n^{[x]}(1)$ *for* $n \geq 2$.

$\mathcal{PR}$ corresponds to functions computable using *loop programs.* $\mathcal{E}$ corresponds to computable functions bounded by some iterate of the exponential function [29,23].

The following facts are known:

Proposition 2 ([29,23]).

- $\mathcal{E}_3 = \mathcal{E} \subsetneq \mathcal{PR} \subsetneq \mathcal{R}ec$
- $\mathcal{E}_n \subsetneq \mathcal{E}_{n+1}$ *for* $n \geq 3$.
- $\mathcal{PR} = \bigcup_i \mathcal{E}_i$

Previous classes can also be related to complexity classes. If TIME(t) and SPACE(t) denote the classes of functions that are computable with time and space t, then:

Proposition 3 ([29,23]). *For all $n \geq 3$,*

- $\mathcal{E}_n = \mathrm{TIME}(\mathcal{E}_n) = \mathrm{SPACE}(\mathcal{E}_n)$,
- $\mathcal{PR} = \mathrm{TIME}(\mathcal{PR}) = \mathrm{SPACE}(\mathcal{PR})$.

In classical computability, more general objects than functions over the integers can be considered, in particular functionals, i.e. functions $\Phi : (\mathbb{N}^{\mathbb{N}})^m \times \mathbb{N}^k \to \mathbb{N}^l$. A functional will be said to be *elementary* (respectively. $\mathcal{E}_n$, *primitive recursive, recursive*) when it belongs to the corresponding[4] class.

[4] Formally, a function f over the integers can be considered as functional $\overline{f} : (V_1, \ldots, V_m, \overrightarrow{n}) \mapsto f(\overrightarrow{n})$. Similarly, an operator Op on functions $f_1, \ldots, f_m$ over the integers can be extended to $\overline{Op}(F_1, \ldots, F_m) : (V_1, \ldots, V_m, \overrightarrow{n}) \mapsto Op(F_1(V_1, \ldots, V_m, .), \ldots, F_m(V_1, \ldots, V_m, .))(\overrightarrow{n})$. We will still (abusively) denote by $[f_1, \ldots, f_p; O_1, \ldots, O_q]$ for the smallest class of functionals that contains basic functions $\overline{f_1}, \ldots, \overline{f_p}$, plus the functionals $Map_i : (V_1, \ldots, V_m, n) \to (V_i)_n$, the nth element of sequence V_i, and which is closed by the operators $\overline{O_1}, \ldots, \overline{O_q}$. For example, a functional will be said to be elementary iff it belongs to $\mathcal{E} = [Map, \overline{0}, \overline{S}, \overline{U}, \overline{+}, \overline{\ominus}; \overline{\mathrm{COMP}}, \overline{\mathrm{BSUM}}, \overline{\mathrm{BPROD}}]$.

4 Computable Analysis

The idea sustaining *Computable analysis*, also called *recursive analysis*, is to define computable functions over real numbers by considering functionals over fast-converging sequences of rationals [33,18,15,34].

Formally, assume that a representation of rational numbers by integers is fixed[5]: let $\nu_{\mathbb{Q}}(r)$ be the rational represented by integer r. A product $\mathcal{C} = [a_1, b_1] \times \ldots \times [a_k, b_k]$ of compact intervals with rational endpoints can be encoded by an integer $\nu(\mathcal{C})$ encoding a list $\langle r_1, r'_1, \ldots, r_k, r'_k \rangle$ with $\nu_{\mathbb{Q}}(r_i) = a_i$, $\nu_{\mathbb{Q}}(r'_i) = b_i$.

A sequence of integers $(x_i) \in \mathbb{N}^{\mathbb{N}}$ *represents a real number* x if it converges quickly toward x (denoted by $(x_i) \rightsquigarrow x$) in the following sense:

$$\forall i, |\nu_{\mathbb{Q}}(x_i) - x| < \exp(-i).$$

For $X = ((x_1), \ldots, (x_k)) \in (\mathbb{N}^{\mathbb{N}})^k$, $\overrightarrow{x} = (x_1, \ldots, x_k) \in \mathbb{R}^k$, we write $X \rightsquigarrow \overrightarrow{x}$ for $(x_i) \rightsquigarrow x_i$ for $i = 1, \ldots, k$.

Definition 1 (Recursive analysis). *A function $f : \mathcal{D} \to \mathbb{R}$, where $\mathcal{D} \subset \mathbb{R}^k$ is a product of compact intervals with rational endpoints, is said to be computable (in the sense of recursive analysis) if there exists a recursive functional $\phi : (\mathbb{N}^{\mathbb{N}})^k \times \mathbb{N} \to \mathbb{N}$ such that for all $\overrightarrow{x} \in \mathcal{D}$, for all $X \in (\mathbb{N}^{\mathbb{N}})^k$, we have $(\phi(X, j))_j \rightsquigarrow f(\overrightarrow{x})$ whenever $X \rightsquigarrow \overrightarrow{x}$.*

A function $f : \mathcal{D} \to \mathbb{R}$, where $\mathcal{D} \subset \mathbb{R}^k$ is not necessarily compact, is said to be computable if there exists a recursive functional $\phi : (\mathbb{N}^{\mathbb{N}})^k \times \mathbb{N}^2 \to \mathbb{N}$ such that for all non-empty product $\mathcal{C}$ of compact intervals with rational endpoints included in $\mathcal{D}$, $\forall \overrightarrow{x} \in \mathcal{C}$, for all $X \in (\mathbb{N}^{\mathbb{N}})^k$, we have $(\phi(X, \nu(\mathcal{C}), j))_j \rightsquigarrow f(\overrightarrow{x})$ whenever $X \rightsquigarrow \overrightarrow{x}$.

A function $f : \mathcal{D} \to \mathbb{R}^l$, with $l > 1$, is said to be computable if all its projections are.

A function f will be said to be *elementarily* (respectively $\mathcal{E}_n$) computable whenever the corresponding functional ϕ is. The class of elementarily (respectively $\mathcal{E}_n$) computable functions over the reals will be denoted by $\mathcal{E}(\mathbb{R})$ (resp. $\mathcal{E}_n(\mathbb{R})$).

Elementarily computable functions have been discussed in [35]. Observing that classical proofs for computable functions (see *e.g.* [34]) use only elementary functionals one can state:

Proposition 4. *Functions $+$, $-$, $\times$, e^x, $\sin(x)$, $\cos(x)$, $1/x$ are elementarily computable in the sense of recursive analysis.*

The following result is also well-known:

Proposition 5 (see e.g. [34]). *All (elementarily) computable functions in the sense of recursive analysis are continuous.*

[5] We will assume that in this representation, the basic functions on rationals $+$, $-$, $\times$, $/$ are elementarily computable.

Actually, one can go further: adapting to the elementary case the classical statements and proofs of recursive analysis (see *e.g.* [34]), one can state that elementarily computable functions are uniformly continuous on all compact subsets of their domains with an elementarily computable modulus of continuity.

Definition 2. *A* modulus of continuity *of a function $f : \mathcal{D} \to \mathbb{R}^l$ defined over a compact domain is a function $M : \mathbb{N} \to \mathbb{N}$ such that for all $i \in \mathbb{N}$, for all x, y,*

$$\|x - y\| < \exp(-M(i)) \Rightarrow \|f(x) - f(y)\| < \exp(-i).$$

Proposition 6. *If $f \in \mathcal{E}(\mathbb{R})$ is defined over a compact domain, then f has an elementarily computable modulus of continuity.*

Proof. We sketch the proof for $f : \mathcal{D} \subset \mathbb{R} \to \mathbb{R}$. The general case is easy to obtain. Function f is computed by elementary functional $\phi : \mathbb{N}^{\mathbb{N}} \times \mathbb{N} \to \mathbb{N}^{\mathbb{N}}$. ϕ can be understood as being a sequence of elementary functions φ_i such that, $\forall x$, $\forall X \rightsquigarrow x, \forall i \in \mathbb{N}, |\nu_{\mathbb{Q}}(\varphi_i(X)) - f(x)| < \exp(-i)$. Each of these φ_i is an elementary function that returns a result in finite time. Hence, it must return a result in a time bounded by an elementary function ψ_i (an iterate of the exponential function). That implies that $\varphi_i(X)$ only depends on the ψ_i first terms of X. If $\|x - y\| < \exp(-\psi_i)$, since we can find $X \rightsquigarrow x$, and $Y \rightsquigarrow y$ that coincide on the ψ_i first terms, we must have $\|f(x) - f(y)\| < 2\exp(-i)$. That means that $\psi(i) = \psi_{i+1} + 1$ is a elementarily computable modulus of continuity for f.

When f is (elementarily) computable, then its derivative f' is not necessarily computable. However, this holds for functions of class $\mathcal{C}^2$ over a compact domain (we are still adapting to the elementary case the classical proofs of recursive analysis: see *e.g.* [34]):

Lemma 3. *Let $f : \mathcal{D} \subset \mathbb{R}^k \to \mathbb{R}^l$ be a function of class $\mathcal{C}^2$ defined over compact domain $\mathcal{D}$.*

If f is elementarily computable, then its partial derivatives are.

Proof. We give the proof for a function f defined on interval $[0, 1]$ to $\mathbb{R}$. The general case is easy to obtain.

Since f'' is continuous on a compact set, f'' is bounded by some constant M. By mean value theorem, we have $|f'(x) - f'(y)| \leq M|x - y|$ for all x, y.

Given $x \in [0, 1]$, and $i \in \mathbb{N}$, an approximation z of $f'(x)$ at precision $\exp(-i)$ can be computed as follows: compute n with $M \exp(-n) \leq \exp(-i)/2$ and $\exp(-n) \leq 1/2$. Compute y_1 a rational at most $\exp(-i)/2$ far from $f(x)$, and y_2 a rational at most $\exp(-i)/2$ far from $f(x+\exp(-n))$. Take $z = (y_1 - y_2)/\exp(-n)$.

This is indeed a value at most $\exp(-i)$ far from $f'(x)$ since by mean value theorem there exists $\chi \in [x, x + \exp(-n)]$ such that $f'(\chi_j) = \frac{f(x+\exp(-n))-f(x)}{\exp(-n)}$. Now

$$\begin{aligned}
|z - f'(x)| &\leq \tfrac{|y_1 - f(x)|}{\exp(-n)} + \tfrac{|y_2 - f(x+\exp(-n))|}{\exp(-n)} + |\tfrac{f(x+\exp(-n)) - f(x)}{\exp(-n)} - f'(x)| \\
&\leq exp(-i)\exp(n)/2 + \exp(-i)\exp(n)/2 + |f'(\chi_j) - f'(x)| \\
&\leq \exp(-i)\exp(n) + M\exp(-n) \\
&\leq \exp(-i)/2 + \exp(-i)/2 \\
&\leq \exp(-i).
\end{aligned}$$

5 Real-recursive and recursive functions

Following the original ideas from [20], but observing that the minimization schema considered in [20] is the source of many technical problems, Campagnolo proposed in his PhD dissertation [11] not to consider classes of functions over the reals defined in analogy with the full class of recursive functions, but with subclasses. Indeed, the considered classes are built in analogy with class of elementary functions and the classes of the Grzegorczyk hierarchy . Furthermore, Campagnolo proposed to restrict the integration schema to a simpler (and better defined) linear integration schemata LI [11,10].

We call *real extension of a function* $f : \mathbb{N}^k \to \mathbb{N}^l$ a function $\tilde{f}$ from $\mathbb{R}^k$ to $\mathbb{R}^l$ whose restriction to $\mathbb{N}^k$ is f.

Definition 3 ([11,10]). *Let $\mathcal{L}$ and $\mathcal{L}_n$ be the classes of functions $f : \mathbb{R}^k \to \mathbb{R}^l$, for some $k, l \in \mathbb{N}$, defined by*

$$\mathcal{L} = [0, 1, -1, \pi, U, \theta_3; \mathrm{COMP}, \mathrm{LI}]$$

and

$$\mathcal{L}_n = [0, 1, -1, \pi, U, \theta_3, \overline{E}_{n-1}; \mathrm{COMP}, \mathrm{LI}]$$

where the base functions 0, 1, -1, π, $(U_i^m)_{i,m\in\mathbb{N}}$, θ_3, $\overline{E}_n$ *and the schemata* COMP *and* LI *are defined as follows:*

1. $0, 1, -1, \pi$ *are the corresponding constant functions;* $U_i^m : \mathbb{R}^m \to \mathbb{R}$ *are, as in the classical settings, projections:* $U_i^m : (x_1, \ldots, x_m) \mapsto x_i$;
2. $\theta_3 : \mathbb{R} \to \mathbb{R}$ *is defined as* $\theta_3 : x \mapsto x^3$ *if* $x \geq 0, 0$ *otherwise.*
3. $\overline{E}_n$*: for* $n \geq 3$*, let* $\overline{E}_n$ *denote a monotone real extension of the function* $\exp_n$ *over the integers defined inductively by* $\exp_2(x) = 2^x$, $\exp_{i+1}(x) = \exp_i^{[x]}(1)$.
4. COMP*: composition is defined as in the classical settings: Given* f *and* g, $h = \mathrm{COMP}(f, g)$ *is the function verifying* $h(\overrightarrow{x}) = g(f(\overrightarrow{x}))$;
5. LI*: linear integration. From* g *and* h, $\mathrm{LI}(g, h)$ *is the maximal solution of the linear differential equation* $\frac{\partial f}{\partial y}(\overrightarrow{x}, y) = h(\overrightarrow{x}, y) f(\overrightarrow{x}, y)$ *with* $f(\overrightarrow{x}, 0) = g(\overrightarrow{x})$.
   In this schema, if g *goes to* $\mathbb{R}^n$, $f = \mathrm{LI}(g, h)$ *also goes to* $\mathbb{R}^n$ *and* $h(\overrightarrow{x}, y)$ *is a* $n \times n$ *matrix with elements in* $\mathcal{L}$.

Lemma 4. *These classes contain functions* $id : x \mapsto x$, $\sin$, $\cos$, $\exp$, $+, \times$, $x \mapsto r$ *for all rational* r*, as well as for all* $f \in \mathcal{L}$*, or* $f \in \mathcal{L}^*$*, its primitive function* F *equal to* $\overrightarrow{0}$ *at* $\overrightarrow{0}$*, denoted by* $\int(f)$.

Proof. Indeed, $\int(f)$ can be defined by $\begin{pmatrix} F \\ 1 \end{pmatrix} = \mathrm{LI}\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 & f \\ 0 & 0 \end{bmatrix}\right)$.

Function id is given by $\int(1)$.

Function $\Theta : t \mapsto (\sin(t), \cos(t))$ can be defined by $\mathrm{LI}\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\right)$.
Project this function on each of its two variables to get sinus and cosinus function.

Function exp is given by $\mathrm{LI}(0,1)$.

Addition is given by $x+0=x$, $\frac{\partial x+y}{\partial y}=1$. Multiplication is given by $x\times 0=0$, $\frac{\partial x\times y}{\partial y}=x$.

Given $p,q\in\mathbb{N}$ with $q>0$, Function $x\mapsto p$, is $1+1+\ldots+1$, function $x\mapsto x^{q-1}$ is $x\times\ldots\times x$, and $p\times\int(x\mapsto x^{q-1})$ is $x\mapsto px^q/q$ whose value in 1 is p/q.

However, non total functions like $x\mapsto 1/x$ can not belong to the class since all functions from $\mathcal{L}$ are total:

Proposition 7 ([10]). *All functions from $\mathcal{L}$ and $\mathcal{L}_n$ are continuous, defined everywhere, and of class $\mathcal{C}^2$.*

The previous classes can be partially related to classes $\mathcal{E}$, $\mathcal{E}_n$ over integers and to classes $\mathcal{E}(\mathbb{R})$ and $\mathcal{E}_n(\mathbb{R})$ over real numbers. Indeed, in order to compare functions over the reals with functions over the integers, we introduce the following notation: given some class $\mathcal{C}$ of functions from $\mathbb{R}^k$ to $\mathbb{R}^l$, we write $\mathrm{DP}(\mathcal{C})$ (DP stands for discrete part) for the class of functions from $\mathbb{N}^k$ to $\mathbb{N}^l$ which have a real extension in $\mathcal{C}$.

One main contribution of [10] is:

Proposition 8 ([11,10]).

- $\mathrm{DP}(\mathcal{L})=\mathcal{E}$;
- $\mathrm{DP}(\mathcal{L}_\mathrm{n})=\mathcal{E}_\mathrm{n}$.

Actually, stronger inclusions were proved in [11,10]:

Proposition 9 ([11,10]).

- $\mathcal{L}\subset\mathcal{E}(\mathbb{R})$.
- $\mathcal{L}_n\subset\mathcal{E}_n(\mathbb{R})$.

However there is no hope to get the other inclusion: these inclusions are strict. Indeed, $x\mapsto 1/x$ is elementarily computable while Proposition 7 says that all functions from $\mathcal{L}$ are defined everywhere. A similar argument works for $\mathcal{E}_n(\mathbb{R})$. We conjecture the inclusions to be strict even when restricting to total functions.

Remark 2. Let θ_k be the function defined by $\theta_k(x)=\begin{cases}x^k & \text{if } x>0\\ 0 & \text{otherwise}\end{cases}$.

If one replace θ_3 by θ_k for a $k>3$ in the definitions of $\mathcal{L}$ and $\mathcal{L}_n$, the classes $\mathcal{L}$ and $\mathcal{L}_n$ may differ from previous ones.

However:

- Propositions 8 and 9 still hold for the obtained classes
- Proposition 7 is changed into "All functions from $\mathcal{L}$ and $\mathcal{L}_n$, are continuous, defined everywhere, and of class $\mathcal{C}^{k-1}$"

Remark 3. Note that all base functions except θ_3 (and the θ_k) are analytic, and that all previous schemes preserve analyticity: in other words, the use of such a function θ_k is necessary in order to be able not to consider only analytic functions.

6 Real-recursive and recursive functions revisited

We now propose to consider new classes of functions that we will prove to correspond precisely to $\mathcal{E}(\mathbb{R})$ and $\mathcal{E}_n(\mathbb{R})$.

First, we modify a little bit the composition schema.

The necessity of doing so follows from following Lemma which can be easily established (and provide another argument proving that $\mathcal{L}$ can not be equal to $\mathcal{E}(\mathbb{R})$):

Lemma 5. *Elementarily computable functions are not stable by composition*[6].

Our new schema is:

Definition 4 (COMP schema). *Given f,g, if there is a product of closed intervals*[7] *C with rational or infinite endpoints with $Range(f) \subset C \subset Domain(g)$, then function* $\mathrm{COMP}(f, g)$ *is defined. It is defined by* $\mathrm{COMP}(f, g) : \overrightarrow{x} \mapsto g(f(\overrightarrow{x}))$ *on all $\overrightarrow{x}$ where $f(\overrightarrow{x})$ and $g(f(\overrightarrow{x}))$ exist.*

Now, we suggest to add a limit operator.

Remark 4. The idea of adding a limit operator has already been investigated in papers like [21,22]. However, since we are interested in $\mathbb{R}$-sub-recursive functions, and not to build a whole hierarchy above recursive functions as in [21,22], our limit schema will not be as general: as Campagnolo's LI is a restrained version of Moore's integration operator, our LIM may be seen as a restrained version of the operators of [21,22].

The conditions we impose on LIM are inspired from Lemma 1: a polynomial β over $\mathbb{R}$ is a function of the form $\beta : \mathbb{R} \to \mathbb{R}$, $\beta : x \mapsto \sum_{i=0}^{n} a_i x^i$ for some $a_0, \dots, a_n \in \mathbb{R}$. A polynomial β over $\mathbb{R}^{k+1}$ is a function of the form $\beta : \mathbb{R}^{k+1} \to \mathbb{R}$, $\beta : \overrightarrow{x} \mapsto \sum_{i=0}^{n_{k+1}} a_i(x_1, \dots, x_k) x_{k+1}^i$ for some $a_0, \dots, a_n$ polynomial over $\mathbb{R}^k$.

Definition 5 (LIM **schema).** *Let $f : \mathbb{R} \times \mathcal{D} \subset \mathbb{R}^{k+1} \to \mathbb{R}^l$, and $\beta : \mathcal{D} \to \mathbb{R}$ a polynomial with the following hypothesis: there exists a constant K such that for all t, $\overrightarrow{x} = (x_1, \dots, x_k)$, $\|\frac{\partial f}{\partial t}(t, \overrightarrow{x})\| \leq K \exp(-t\beta(\overrightarrow{x}))$, $\frac{\partial^2 f}{\partial t \partial x_i}(t, x_i)$ exists for all $1 \leq i \leq k$, and $\|\frac{\partial^2 f}{\partial t \partial x_i}(t, \overrightarrow{x})\| \leq K \exp(-t\beta(\overrightarrow{x}))$.*

Then, for every product of intervals $I \subset \mathbb{R}^k$ on which $\beta(\overrightarrow{x}) > 0$, $F =$ $\mathrm{LIM}(f, \beta)$ is defined as the function $F : I \to \mathbb{R}$, with $F(\overrightarrow{x}) = \lim_{t \to +\infty} f(t, \overrightarrow{x})$, under the condition that it is of class[8] *$\mathcal{C}^2$.*

We are ready to define our classes:

Definition 6 (Classes $\mathcal{L}^*$, $\mathcal{L}_n^*$). *The class $\mathcal{L}^*$, and $\mathcal{L}_n^*$, for $n \geq 3$, of functions from $\mathbb{R}^k$ to $\mathbb{R}^l$, for $k, l \in \mathbb{N}$, are following classes:*

[6] The proof uses non-total functions. Total elementarily computable functions can be shown stable by composition.

[7] That can be $\mathbb{R}^k$ when g is total.

[8] If f is of class $\mathcal{C}^1$, function F exists and is at least of class $\mathcal{C}^1$ by Lemma 1.

– $\mathcal{L}^* = [0, 1, -1, U, \theta_3; \mathrm{COMP}, \mathrm{LI}, \mathrm{LIM}]$.
– $\mathcal{L}^*_n = [0, 1, -1, U, \theta_3, \overline{E}_{n-1}; \mathrm{COMP}, \mathrm{LI}, \mathrm{LIM}]$.

Remark 5. Previous classes can easily be shown stable by the primitive operator that sends a function f to its primitive $\int(f)$ equal to $\overrightarrow{0}$ at $\overrightarrow{0}$.

Indeed, $\int(f)$ can still be defined by $\begin{pmatrix} F \\ 1 \end{pmatrix} = \mathrm{LI}\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 & f \\ 0 & 0 \end{bmatrix}\right)$.

Remark 6. Unlike classes from previous sections, class $\mathcal{L}^*$ also includes some non-total functions.

In particular the function $\frac{1}{x} : \begin{cases} \mathbb{R}^{>0} \to \mathbb{R} \\ x \mapsto \frac{1}{x} \end{cases}$. Indeed, $E(t, x) = \int(\exp(-tx))$ is such that $E(t, x) = \begin{cases} \frac{(1-\exp(-tx))}{x} & \text{for } x \neq 0 \\ t & \text{for } x = 0 \end{cases}$ (of class $\mathcal{C}^k$ for all k). Now $\frac{1}{x} = \mathrm{LIM}(E, id)$.

Our classes are supersets of previous classes:

Proposition 10. *$\mathcal{L} \subsetneq \mathcal{L}^*$, $\mathcal{L}_n \subsetneq \mathcal{L}^*_n$ for all $n \geq 3$.*

Proof. The function $x \mapsto \pi$ is actually in $\mathcal{L}^*$. Indeed, from $x \mapsto \frac{1}{1+x^2}$ in the class , we have $\arctan x = \int(\frac{1}{1+x^2})$, and $\pi = 4\arctan(1)$. Observing that our composition schema for total functions subsumes the composition schema of class $\mathcal{L}$, the result follows.

The main results of this paper are the following (proved in following two sections):

Theorem 1 (Characterization of $\mathcal{E}(\mathbb{R})$). *Let $f : \mathcal{D} \subset \mathbb{R}^k \to \mathbb{R}^l$ be some function over the reals of class $\mathcal{C}^2$, with $\mathcal{D}$ product of compact intervals with rational endpoints.*

f is in $\mathcal{E}(\mathbb{R})$ iff it belongs to $\mathcal{L}^*$.

Theorem 2 (Characterization of $\mathcal{E}_n(\mathbb{R})$). *Let $f : \mathcal{D} \subset \mathbb{R}^k \to \mathbb{R}^l$ be some function over the reals of class $\mathcal{C}^2$, with $\mathcal{D}$ product of compact intervals with rational endpoints. Let $n \geq 3$.*

f is in $\mathcal{E}_n(\mathbb{R})$ iff it belongs to $\mathcal{L}^*_n$.

Remark 7. If we replace θ_3 by θ_k for a $k \geq 3$ in the definitions of $\mathcal{L}^*$ and $\mathcal{L}^*_n$, and impose the result of a LIM operation to be of class $\mathcal{C}^{k-1}$ in Definition 5 (instead of $\mathcal{C}^2$), the classes $\mathcal{L}^*$ and $\mathcal{L}^*_n$ may differ. However, we have almost the same theorems for the corresponding classes: replace $\mathcal{C}^2$ by $\mathcal{C}^{k-1}$ in the statements of the theorems.

7 Upper bounds

We now prove the upper bound $\mathcal{L}^* \subset \mathcal{E}(\mathbb{R})$. As one may expect, this direction of the proof has many similarities with the proof $\mathcal{L} \subset \mathcal{E}$ in [11,10]: main differences lie in the presence of non-total functions and of schema LIM.

We first discuss the domain of the considered functions.

Lemma 6. *All functions from $\mathcal{L}^*$ are of class $\mathcal{C}^2$ and defined on a domain of the form $I_1 \times I_2 \ldots \times I_k$ where each I_i is an interval.*

Proof. By structural induction

- This is clear for basic functions (1, 0, -1, U, and θ_3).
- Composition preserves this property.
- Linear differential equations preserve class $\mathcal{C}^2$ [1,12]. They also preserve the domain property from Lemma 2 [1,12].
- If $g = \mathrm{LIM}(f, \beta)$, from definition of LIM schema, this is clear.

We propose to introduce the following notation: given $a \in \mathbb{R}$, let ρ_a be the function $x \mapsto \frac{1}{x-a}$. Let $\rho_{+\infty}$ and $\rho_{-\infty}$ be the function identity $x \mapsto x$.

Given I real interval with bounds $a, b \in \mathbb{R} \cup \{-\infty, +\infty\}$, $\rho_I(x) = |\rho_a(x)| + |\rho_b(x)|$. For $\mathcal{D} = I_1 \times I_2 \ldots \times I_k$, let $\rho_{\mathcal{D}}(x) = \rho_{I_1}(U_1^k(x)) + \ldots + \rho_{I_k}(U_k^k(x))$. In any case, $\rho_{\mathcal{D}}(x)$ is elementarily computable and grows to $+\infty$ when x gets close to a bound of domain $\mathcal{D}$.

The following Lemma is an extension of a Lemma of [11,10].

Lemma 7. *Let $f : \mathcal{D} \subset \mathbb{R}^k \to \mathbb{R}^l$ be a function of $\mathcal{L}^*$. There exist some integer d, and some constants A and B such that for all $\overrightarrow{x} \in \mathcal{D}$, $\|f(\overrightarrow{x})\| \leq A \exp^{[d]}(B \rho_{\mathcal{D}}(\overrightarrow{x}))$. Call the smallest such integer d the degree of f (denoted by $\deg f$). All the partial derivatives of f also have a finite degree.*

Proof. By some elementary algebra and elementary properties of the exponential function, observe that by adjusting constants A, B, it is always possible to assume for all functions f and g, $\deg fg \leq max(\deg f, \deg g)$, and $\deg(f+g) \leq max(\deg f, \deg g)$.

Now, by structural induction:

- 0, 1, -1, U and all their derivatives have degree at most 1.
- $\theta_3(x)$ and its derivative have degree 1.
- The degree of $\mathrm{COMP}(f, g)$ is less than $\deg(f) + \deg(g)$, since $\deg(f \circ g) \leq \deg(f) + \deg(g)$ can easily be established using basic properties of exponential function. By the chain rule, the degree of any of the derivative of the composition $f(g)$ is bounded by $\max_i(\deg \frac{\partial g}{\partial i}, \deg \frac{\partial f}{\partial i} + \deg g)$.
- For $f = \mathrm{LI}(g, h)$ as in Definition 3, Lemma 2 allows us to write

$$\|f(x, y)\| \leq \|g(x)\| \exp(\sup_{\tau \in [0, y]} \|h(x, \tau)\| y).$$

It follows that the degree of f is less than $\max(\deg g, \deg h + 1)$.

The derivative of f relative to y is $h(\overrightarrow{x},y)f(\overrightarrow{x},y)$. Hence its degree is also bounded by $\max(\deg g, \deg h+1)$. By [1,12], we know that the other derivative relative to variable x is solution of linear differential equation $d' = hd + \frac{\partial h}{\partial i} f$ with initial condition $d(x,0) = \frac{\partial g}{\partial x}$. The bound given by Lemma 2 for this linear differential equation allows us to state that the degree of this derivative is less than $\max(\deg \frac{\partial g}{\partial x}, \deg h+1, \deg \frac{\partial h}{\partial x}+1, \deg f+1)$.
- Let $g = \mathrm{LIM}(f,\beta)$ as in Definition 5. By Lemma 1, we know that $g(\overrightarrow{x}) = \lim_{i\to\infty} f(i,\overrightarrow{x})$, g is of class $\mathcal{C}^1$, $\|g(\overrightarrow{x})\| \leq \|f(0,\overrightarrow{x})\| + K/\beta(\overrightarrow{x})$ and $\|\frac{\partial g}{\partial x_i}\| \leq \|\frac{\partial f}{\partial x_i}(0,\overrightarrow{x})\| + K/\beta(\overrightarrow{x})$. Now, the degree of $\frac{1}{\beta(\overrightarrow{x})}$ for any polynomial β can easily be shown to be less than 1. Hence, the degree of g and of $\frac{\partial g}{\partial x}$ is smaller than the degree of f.

We are ready to prove the upper bound.

Proposition 11. $\mathcal{L}^* \subseteq \mathcal{E}(\mathbb{R})$.

Proof. By structural induction:

- The basic functions $0, 1, -1, U, \theta_3$ are easily shown elementarily computable.
- When $h = \mathrm{COMP}(f,g)$, f and g elementarily computable, then h is also elementarily computable: indeed, there exists some closed set F with $Range(f) \subset F \subset Domain(g)$. Adapting the constructions in [34], given a product of compact intervals $\mathcal{C}$ with rational endpoints included in $Domain(f)$, we can compute elementarily a product of compact intervals $\mathcal{C}'$ with rational endpoints with $f(\mathcal{C}) \subset \mathcal{C}'$. Now, for $x \in \mathcal{C}$, compose the functional that computes g on $\mathcal{C}' \cap F$ with the one that computes f on $\mathcal{C}$.
- Let $g = \mathrm{LIM}(f,\beta)$, with f computed by elementary functional ϕ. We give the proof for f defined on $\mathbb{R} \times \mathcal{C}$ to $\mathbb{R}$ where $\mathcal{C}$ is a compact interval of $\mathbb{R}$. The general case is easy to obtain.
  Let $x \in \mathbb{R}$, with $\beta(x) > 0$. Since $\beta(x)$ is a polynomial, $1/\beta(x)$ can be bounded elementarily by some computable integer N in some computable neighborhood of x.
  Let $(x_n) \rightsquigarrow x$. For all $i, j \in \mathbb{N}$, if we write (i) for the constant sequence $k \mapsto i$, we have $|\nu_{\mathbb{Q}}(\phi((i),(x_n),j)) - f(i,x)| < \exp(-j)$.
  By Lemma 1, we have $|f(i,x) - g(x)| \leq \frac{K\exp(-\beta(x)i)}{\beta(x)} \leq KN\exp(-\beta(x)i)$.
  Hence, $|\nu_{\mathbb{Q}}(\phi((i),(x_n),j)) - g(x)| < \exp(-j) + KN\exp(-\beta(x)i)$.
  If we take $j' = j+1$, $i' = N(j+1+\lceil\ln(KN)\rceil)$, we have $\exp(-j') \leq \exp(-j)/2$, and $KN\exp(-\beta(x)i') \leq \exp(-j)/2$. Hence g is computed by the functional $\psi : ((x_n),j) \mapsto \phi((N(j+1+\lceil\ln(KN)\rceil)),(x_n),j+1)$.
- Let $f = \mathrm{LI}(g,h)$. We give the proof for $g : [0,1] \to \mathbb{R}$ and $h : [0,1]\times[c,d] \to \mathbb{R}$. The general case is easy to obtain.
  This proof is copied from [11,10]. The idea is that, to find ϕ elementary computing f, one uses a numeric integration algorithm (Euler's Method).
  First, let us note that f is twice differentiable with respect to its second variable since its derivative is the product of f and h that are differentiable.
  To compute $f(x,y)$, we will slice $[0,y]$ into segments of length λ and compute approximations of $f(x,\tau_i)$ for τ_i multiple of λ.

$h \in \mathcal{E}(\mathbb{R})$. Let ϕ_h computing h. Let $(\phi) \rightsquigarrow (x, \tau_i)$. Let us define $\omega_i = \frac{(\phi_h(\phi))_n}{n+1}$ for n to be chosen.
$f \in \mathcal{E}(\mathbb{R})$. Let ϕ_g computing g. Let $(\phi_x) \rightsquigarrow x$. We will approach $f(x, \tau_i)$ by ψ_i defined by

$$\psi_0 = \frac{(\phi_g(\phi_x))_m}{m+1}$$

$$\psi_{i+1} = \psi_i + \lambda \psi_i \omega_i$$

Let us now compute the error induced by our approximation. Let $\varepsilon_i = f(x, \tau_i) - \psi_i$.
$\forall i, \exists \chi \in [\tau_i, \tau_{i+1}]; f(x, \tau_{i+1}) = f(x, \tau_i) + \lambda f(x, \tau_i) h(x, \tau_i) + \frac{\lambda^2}{2} \frac{\partial f}{\partial y}(x, \chi)$.

$$\begin{aligned}
\varepsilon_{i+1} &= f(x, \tau_i) - \psi_i + \lambda f(x, \tau_i) h(x, \tau_i) - \lambda \psi_i \omega_i + \tfrac{\lambda^2}{2} \tfrac{\partial f}{\partial y}(x, \chi) \\
|\varepsilon_{i+1}| &\leqslant |\varepsilon_i| + |\lambda h(x, \tau_i)(f(x, \tau_i) - \psi_i)| + |\lambda \psi_i(\omega_i - h(x, \tau_i))| + |\tfrac{\lambda^2}{2} \tfrac{\partial f}{\partial y}(x, \chi)| \\
&\leqslant |\varepsilon_i| \times |1 + \lambda h(x, \tau_i)| + \lambda \psi_i |\omega_i - h(x, \tau_i)| + \tfrac{\lambda^2}{2} \beta \\
&< |\varepsilon_i| \times |1 + \lambda h(x, \tau_i)| + \lambda \psi_i \tfrac{1}{n+1} + \tfrac{\lambda^2}{2} \beta
\end{aligned}$$

With $\beta = \max_{\chi \in [0,y]}(\frac{\partial f}{\partial y}(x, \chi))$.

$$\varepsilon_{i+1} < |\varepsilon_i| \times |1 + \lambda \overline{y}| + \tfrac{\lambda}{n+1} \overline{y} + \tfrac{\lambda^2}{2} \beta$$

With $\overline{y}$ set as a bound for h that can be elementarily computed as shown by the preceding lemma.
Some little algebra shows then

$$\begin{aligned}
|\varepsilon_i| &< |\varepsilon_0| \left[1 + \lambda \overline{y}\right]^i + \left(\lambda \overline{y} \tfrac{1}{n+1} + \tfrac{\lambda^2}{2} \beta\right) \tfrac{(1+\lambda \overline{y})^i - 1}{\lambda \overline{y}} \\
&< \left[\tfrac{1}{m+1} + \tfrac{\lambda \beta}{2 \overline{y}} + \tfrac{\psi_i}{\overline{y}(n+1)}\right] \exp(\lambda i \overline{y}) \\
&< \left[\tfrac{1}{m+1} + \tfrac{1}{n+1} + \tfrac{\lambda \beta}{2 \overline{y}}\right] \exp(\lambda i \overline{y})
\end{aligned}$$

So, if we choose m, n, and i adequately (this choice can be made elementarily), we can make the error as little as wanted. This proves that f is elementarily computable and terminates our proof.

This ends the proof.

Replacing in previous proofs the bounds of Lemma 7 by bounds of type $\|f(\overrightarrow{x})\| \leq A \overline{E}_{n-1}^{[d]}(B \rho_{\mathcal{D}}(\overrightarrow{x}))$, one can also obtain:

Proposition 12. $\forall n \geq 3$, $\mathcal{L}_n^* \subseteq \mathcal{E}_n(\mathbb{R})$.

8 Lower bounds

We will now consider the opposite inclusion: $\mathcal{E}(\mathbb{R}) \subseteq \mathcal{L}^*$, proved for functions of class $\mathcal{C}^2$ on compact domains with rational endpoints.

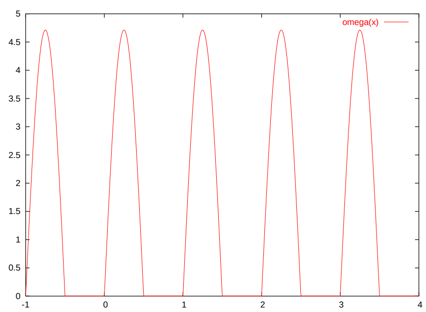
Let $\epsilon > 0$ be some real. We write $\mathbb{N}\epsilon$ for the set of reals of the form $i\epsilon$ for some integer i. Given $y \in \mathbb{R}$, write $\lfloor y \rfloor_\epsilon$ for the unique $j\epsilon$ with j integer and $y \in [j\epsilon, j\epsilon + \epsilon)$.

Lemma 8. *Let $\epsilon : \mathbb{R} \to \mathbb{R}$ be some decreasing elementarily computable function, with $\epsilon(x) > 0$ for all x and going to 0 when x goes to $+\infty$. Write ϵ_i for $\epsilon(\lfloor i \rfloor)$.*

Given $f : \mathbb{R}^2 \to \mathbb{R}^l$ in $\mathcal{L}^*$, there exists $F : \mathbb{R}^2 \to \mathbb{R}^l$ in $\mathcal{L}^*$ with the following properties:

- *For all $i \in \mathbb{N}$, $x \in \mathbb{N}\epsilon_i$, $F(i,x) = f(i,x)$*
- *For all $i \in \mathbb{N}$, $x \in \mathbb{R}$, $\|F(i,x) - f(i, \lfloor x \rfloor_{\epsilon_i})\| \leq \|f(i, \lfloor x \rfloor_{\epsilon_i} + \epsilon_i) - f(i, \lfloor x \rfloor_{\epsilon_i})\|$*
- *For all $i \in \mathbb{R}$, $x \in \mathbb{R}$, $\|\frac{\partial F}{\partial i}(i,x)\| \leq 5\|f(\lfloor i+1 \rfloor, \lfloor x \rfloor_{\epsilon_i}) - f(\lfloor i \rfloor, \lfloor x \rfloor_{\epsilon_i})\| + 25\|f(\lfloor i \rfloor, \lfloor x \rfloor_{\epsilon_i} + \epsilon_i) - f(\lfloor i \rfloor, \lfloor x \rfloor_{\epsilon_i})\| + 25\|f(\lfloor i+1 \rfloor, \lfloor x \rfloor_{\epsilon_{i+1}} + \epsilon_{i+1}) - f(\lfloor i+1 \rfloor, \lfloor x \rfloor_{\epsilon_{i+1}})\|$.*

Proof. Let $\zeta = \frac{3\pi}{2}$. Let $\omega : x \mapsto \zeta\theta_3(\sin(2\pi x))$. $\forall i, \int_i^{i+1} \omega = 1$ and ω is equal to 0 on $[i + \frac{1}{2}, i+1]$ for $i \in \mathbb{N}$. Let $\Omega = \int(\omega)$ its primitive, and $int : x \mapsto \Omega(x - \frac{1}{2})$. int is a function similar to the integer part: $\forall i, \forall x \in [i, i + \frac{1}{2}]$, $int(x) = i = \lfloor x \rfloor$. Figure 1 shows graphical representations of ω and int respectively.



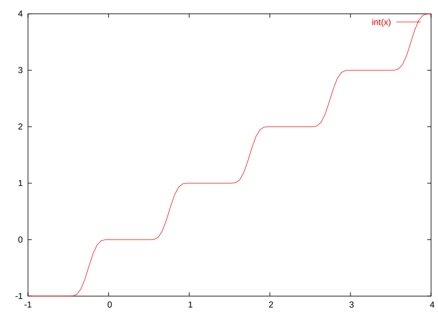


Figure 1. Graphical representations of ω and int.

Let $\Delta(i,x) = f(i, x + \epsilon(i)) - f(i,x)$. For all i,x, we have

$$\begin{aligned} \tfrac{\omega(x/\epsilon(i))}{\epsilon(i)} \Delta(i, \epsilon(i)\ int(x/\epsilon(i))) &= 0 \text{ whenever } x - \lfloor x \rfloor_{\epsilon(i)} \geq \epsilon(i)/2 \\ &= \tfrac{\omega(x/\epsilon(i))}{\epsilon(i)} \Delta(i, \lfloor x \rfloor_{\epsilon(i)}) \text{ otherwise.} \end{aligned}$$

Let G be the solution of the linear differential equation

$$\begin{cases} G(i,0) &= f(0) \\ \frac{\partial G}{\partial x}(i,x) &= \frac{\gamma(x/\epsilon(i))}{\epsilon(i)} \Delta(i, \epsilon(i) int(x/\epsilon(i))) \end{cases}$$

An easy induction on j then shows that $G(i, j\epsilon(i)) = f(i, j\epsilon(i))$ for all $j \in \mathbb{N}$. On $[j\epsilon(i), (j+1)\epsilon(i))$,

$$G(i,x) - f(i, \lfloor x \rfloor_{\epsilon(i)}) = \int_{j\epsilon(i)}^{x-j\epsilon(i)} \frac{\gamma(x/\epsilon(i))}{\epsilon(i)} \Delta(i, \lfloor x \rfloor_{\epsilon(i)}),$$

hence, for all $i \in \mathbb{N}$,

$$\|G(i,x) - f(i, \lfloor x \rfloor_{\epsilon_i})\| \leq \|\Delta(i, \lfloor x \rfloor_{\epsilon_i})\| = \|f(i, \lfloor x \rfloor_{\epsilon_i} + \epsilon_i) - f(i, \lfloor x \rfloor_{\epsilon_i})\|.$$

Now, let $\Delta'(i,x) = G(i+1,x) - G(i,x)$. For all i,x we have

$$\begin{aligned} \omega(i)\Delta'(int(i),x) &= 0 \ \mathit{whenever}\ i - \lfloor i \rfloor \geq 1/2 \\ &= \omega(i)\Delta'(\lfloor i \rfloor, x)\ \mathit{otherwise} \end{aligned}$$

Let F be the solution of linear differential equation

$$\begin{cases} F(0,x) = G(0,x) \\ \frac{\partial F}{\partial i} \quad\ = \omega(i)\Delta'(int(i),x) \end{cases}$$

An easy induction on i shows that $F(i,x) = G(i,x)$ for all integer i, and all $x \in \mathbb{R}$. Hence $F(i,x) = f(i,x)$ for all $i \in \mathbb{N}$, $x \in \mathbb{N}\epsilon_i$ and

$$\|F(i,x) - f(i, \lfloor x \rfloor_{\epsilon_i})\| \leq \|f(i, \lfloor x \rfloor_{\epsilon_i} + \epsilon_i) - f(i, \lfloor x \rfloor_{\epsilon_i})\|$$

for all $i \in \mathbb{N}$, $x \in \mathbb{R}$.

Now, $\frac{\partial F}{\partial i}$ is either 0 or $\omega(i)\Delta'(\lfloor i \rfloor, x) = \omega(i)(G(\lfloor i+1 \rfloor, x) - G(\lfloor i \rfloor, x))$. In any case, it is derivable in x, and hence $\frac{\partial^2 F}{\partial x \partial i}$ is either 0 or $\omega(i)(\frac{\partial G}{\partial x}(\lfloor i+1 \rfloor, x) - \frac{\partial G}{\partial x}(\lfloor i \rfloor, x))$.

When $x \in \mathbb{N}\epsilon_i$, bounding ω by 5 ($\zeta \leq 5$),

$$\|\frac{\partial F}{\partial i}\| \leq 5\|f(\lfloor i+1 \rfloor, x) - f(\lfloor i \rfloor, x)\|.$$

When $x \in \mathbb{R}$,

$$\|\frac{\partial^2 F}{\partial x \partial i}\| \leq \|\frac{\partial G}{\partial x}(\lfloor i+1 \rfloor, x\| + \|\frac{\partial G}{\partial x}(\lfloor i \rfloor, x)\|.$$

The term $\|\frac{\partial G}{\partial x}(\lfloor i \rfloor, x)\|$ can be either 0 or

$$\begin{aligned} 5\|\frac{\omega(x/\epsilon_i)}{\epsilon_i}\Delta(\lfloor i \rfloor, \lfloor x \rfloor_{\epsilon_i})\| &\leq \frac{25}{\epsilon_i}\|\Delta(\lfloor i \rfloor, \lfloor x \rfloor_{\epsilon_i})\| \\ &\leq \frac{25}{\epsilon_i}\|f(\lfloor i \rfloor, \lfloor x \rfloor_{\epsilon_i} + \epsilon_i) - f(\lfloor i \rfloor, \lfloor x \rfloor_{\epsilon_i})\| \end{aligned}.$$

A similar bound holds for the other term, replacing i by $i+1$.

Using mean value theorem,

$$\begin{array}{ll} \|\frac{\partial F}{\partial i}(i,x)\| & \leq \|\frac{\partial F}{\partial i}(i,\lfloor x\rfloor_{\epsilon_i})\| + \|\frac{\partial^2 F}{\partial x\partial i}(i,x)\|(x-\lfloor x\rfloor_{\epsilon_i}) \\ & \leq \|\frac{\partial F}{\partial i}(i,\lfloor x\rfloor_{\epsilon_i})\| + \epsilon(i)\|\frac{\partial^2 F}{\partial x\partial i}(i,x)\| \end{array},$$

which yields the expected bound.

Lemma 9. *If $f : \mathcal{C} \subset \mathbb{R} \to \mathbb{R}$ is defined over a compact interval with rational endpoints containing $\overrightarrow{0}$, is of class $\mathcal{C}^1$ and is elementarily computable, then the primitive $\int(f)$ is in $\mathcal{L}^*$.*

Proof. Let M denote the elementarily computable modulus of continuity of function f. Say $\mathcal{C} = [a,b]$, with $0 \in \mathcal{C}$. Given $i \in \mathbb{N}$, consider $n = M(i)$, and for all j, consider $x_j = j\exp(-n)$, so that for all $x, y \in [x_j, x_{j+1}]$, we have

$$|f(x) - f(y)| \leq exp(-i).$$

For all j, let p_j and q_j two integers such that $p_j \times \exp(-q_j)$ is at most $\exp(-i)$ far from $f(x_j)$. The functions $p_{\mathbb{N}} : \mathbb{N}^2 \to \mathbb{N}$, and $q_{\mathbb{N}} : \mathbb{N}^2 \to \mathbb{N}$ that map (i,j) to corresponding p_j and q_j are elementarily computable.

By Proposition 9, they can be extended to function $p : \mathbb{R}^2 \to \mathbb{R}$ and $q : \mathbb{R}^2 \to \mathbb{R}$ in $\mathcal{L}$. Consider function $g : \mathbb{R} \times \mathcal{C} \to \mathbb{R}$ defined on all $(i,x) \in \mathbb{R} \times \mathcal{C}$ by $g(i,x) = p(i,\exp(n)x)\exp(-q(i,\exp(n)x))$. By construction, for i, j integer, we have

$$g(i,x_j) = p_j \exp(-q_j)$$

Consider the function F given by Lemma 8 for function g and $\epsilon : n \mapsto \exp(-n)$. We have

$$F(i,x_j) = g(i,x_j)$$

and

$$\|g(i,x_j) - f(x_j)\| \leq \exp(-i)$$

for all i, j.

For all integer i, and all $x \in \mathcal{C}$, we have

$$\begin{array}{ll} \|F(i,x) - f(x)\| & \leq \|F(i,x) - F(i,\lfloor x\rfloor_\epsilon)\| + \|F(i,\lfloor x\rfloor_\epsilon - g(i,\lfloor x\rfloor_\epsilon)\| \\ & \quad + \|g(i,\lfloor x\rfloor_\epsilon) - f(\lfloor x\rfloor_\epsilon)\| + \|f(\lfloor x\rfloor_\epsilon) - f(x)\| \\ & \leq \|F(i,\lfloor x\rfloor_\epsilon + \epsilon) - F(i,\lfloor x\rfloor_\epsilon)\| + 0 + \exp(-i) + \exp(-i) \\ & \leq \|g(i,x_{j+1}) - f(x_{j+1})\| + \|g(i,x_j) - f(x_j)\| \\ & \quad + \|f(x_{j+1}) - f(x_j)\| + 2\exp(-i) \\ & \leq 5 \times \exp(-i) \end{array}$$

Consider the function $G : \mathbb{R}^2 \to \mathbb{R}$ defined for all $i, x \in \mathbb{R}$ by the linear differential equation

$$\begin{cases} G(i,0) & = 0 \\ \frac{\partial G}{\partial x}(i,x) & = F(i,x) \end{cases}$$

Hence

$$G(i,x) = \int_0^x F(i,x)dx.$$

For all integer i, we have

$$\|\frac{\partial G}{\partial x}(i,x) - f(x)\| = \|F(i,x) - f(x)\| \leq 5 \times \exp(-i).$$

By mean value theorem on function $G(i,x) - f(x)$, we get

$$\|G(i,x) - \int(f)(x)\| \leq 5 \times \exp(-i) \times (b-a)$$

on $\mathcal{C} = [a,b]$.

Hence, $\int(f)(x)$ is the limit of $G(i,x)$ when i goes to $+\infty$ with integer values. We just need to check that schema LIM can be applied to function G of $\mathcal{L}^*$ to conclude: indeed, the limit of $G(i,x)$ when i goes to $+\infty$ will exist and coincide with this value, i.e. $\int(f)(x)$.

Since $\frac{\partial G}{\partial x} = F$, and hence $\|\frac{\partial^2 G}{\partial i \partial x}\| = \|\frac{\partial F}{\partial i}\|$ and since $\frac{\partial G}{\partial i} = \int_0^x \frac{\partial F}{\partial i}(i,x)dx$ implies

$$\|\frac{\partial G}{\partial i}\| \leq \int_a^b \|\frac{\partial F}{\partial i}\|dx \leq (b-a)\|\frac{\partial F}{\partial i}\|$$

we only need to prove that we can bound $\|\frac{\partial F}{\partial i}\|$ by $K \times \exp(-i)$ for a constant K.

But from Lemma 8, we know that for all i, x,

$$\begin{aligned}\|\tfrac{\partial F}{\partial i}(i,x)\| \leq\ & 5\|g(\lfloor i+1 \rfloor, \lfloor x \rfloor_{\epsilon_i}) - g(\lfloor i \rfloor, \lfloor x \rfloor_{\epsilon_i})\| \\ & +25\|g(\lfloor i \rfloor, \lfloor x \rfloor_{\epsilon_i} + \epsilon_i) - g(\lfloor i \rfloor, \lfloor x \rfloor_{\epsilon_i})\| \\ & +25\|g(\lfloor i+1 \rfloor, \lfloor x \rfloor_{\epsilon_{i+1}} + \epsilon_{i+1}) - g(\lfloor i+1 \rfloor, \lfloor x \rfloor_{\epsilon_{i+1}})\|.\end{aligned}$$

First term can be bounded by $5 \times \exp(-i) + 5 \times \exp(-i) = 10 \times \exp(-i)$.

Second term can be bounded by $25(\|g(\lfloor i \rfloor, \lfloor x \rfloor_{\epsilon_i} + \epsilon_i) - f(\lfloor x \rfloor_{\epsilon_i} + \epsilon_i)\| + \|f(\lfloor x \rfloor_{\epsilon_i} + \epsilon_i) - f(\lfloor x \rfloor_{\epsilon_i})\| + \|g(\lfloor i \rfloor, \lfloor x \rfloor_{\epsilon_i}) - f(\lfloor x \rfloor_{\epsilon_i})\|) \leq 25 \times \exp(-i) + 25 \times \exp(-i) + 25 \times \exp(-i) = 75 \times \exp(-i)$.

Similarly for third term, replacing i by $i+1$.

Hence

$$\|\frac{\partial F}{\partial i}(i,x)\| \leq 160 \times \exp(-i),$$

and so schema LIM can be applied on function G of $\mathcal{L}^*$ to get function $\int(f)$. This ends the proof.

Actually, the previous lemma can easily be extended a little bit to get any primitive:

Lemma 10. *Let h be elementarily computable and defined on 0.*

If $f : \mathcal{C} \subset \mathbb{R} \to \mathbb{R}$ is defined over a compact containing 0, is of class $\mathcal{C}^1$ and is elementarily computable then the primitive f equal to $h(0)$ in 0 is in $\mathcal{L}^*$.

Proof. Replace in previous proof the initial condition $G(i,0) = 0$ of the differential equation defining function G, by $G(i,0) = g(i)$ where $g : \mathbb{R} \to \mathbb{R}$ is a function converging to $h(0)$, obtained by extending a suitably chosen function $g : \mathbb{N} \to \mathbb{N}$.

We are now ready to prove the missing inclusion of Theorem 1.

Proposition 13. *Let $f : \mathcal{D} \subset \mathbb{R}^k \to \mathbb{R}^l$ be some function over the reals of class $\mathcal{C}^2$, with $\mathcal{D}$ product of compact intervals with rational endpoints. If f is $\mathcal{E}(\mathbb{R})$, then it belongs to $\mathcal{L}^*$.*

Proof. Putting together Lemma 3, Lemma 10 applied on f', we obtain this proposition when $k = l = 1$. The case $k > 1$, $l = 1$ can be obtained by adapting the previous arguments to functions of several variables. The case $l > 1$ is immediate since a function is in $\mathcal{L}^*$ if its projections are.

The missing inclusion of Theorem 2 can be proved similarly for all levels $n \geq 3$ of the Grzegorczyk hierarchy.

Proposition 14. *Let $f : \mathcal{D} \subset \mathbb{R}^k \to \mathbb{R}^l$ be some function over the reals of class $\mathcal{C}^2$, with $\mathcal{D}$ product of compact intervals with rational endpoints. If f is $\mathcal{E}_n(\mathbb{R})$, for $n \geq 3$, then it belongs to $\mathcal{L}^*_n$.*

9 Extensions

We have the following corollary:

Corollary 1. *Let $f : \mathcal{D} \subset \mathbb{R}^k \to \mathbb{R}^l$ be some function over the reals of class $\mathcal{C}^\infty$, with $\mathcal{D}$ product of compact intervals with rational endpoints. If f is $\mathcal{E}(\mathbb{R})$, then all its derivative $f^{(n)}$, $n \geq 0$, belongs to $\mathcal{L}^*$.*

Proof. From Lemma 3, for all n, $f^{(n+1)}$ is elementarily computable since it is of class $\mathcal{C}^2$ over a compact domain. Now, for all n, $f^{(n)}(x) \in \mathcal{L}^*$ from Lemma 10 applied on $f^{(n+1)}$.

Similarly:

Corollary 2. *Let $f : \mathcal{D} \subset \mathbb{R}^k \to \mathbb{R}^l$ be some function over the reals of class $\mathcal{C}^\infty$, with $\mathcal{D}$ product of compact intervals with rational endpoints. If f is $\mathcal{E}_n(\mathbb{R})$, then all its derivative $f^{(n)}$, $n \geq 0$, belongs to $\mathcal{L}^*_n$.*

We also have a kind of *normal form theorem*:

Proposition 15. *If constant function π is added to the base functions of $\mathcal{L}^*$ and $\mathcal{L}^*_n$, then every function of $\mathcal{L}^*$ and $\mathcal{L}^*_n$ can be defined using only* 1 *schema* LIM.

Proof. The previous proof shows that to represent a $\mathcal{C}^2$ function that belongs to $\mathcal{E}(\mathbb{R})$, using one LIM is sufficient, if π is considered as base function (in order to have the inclusion $\mathcal{L} \subset \mathcal{L}^*$ and $\mathcal{L}_n \subset \mathcal{L}^*_n$). That means that all functions from $\mathcal{L}^*$ can be written with at most one LIM in that case.

A corollary of this proposition is that composing several LIM schemata is always equivalent to at most one for functions of our classes, if constant function π is considered as a base function. Otherwise, two limits are sufficient.

References

1. V. I. Arnold. *Ordinary Differential Equations.* MIT Press, 1978.
2. E. Asarin and A. Bouajjani. Perturbed Turing machines and hybrid systems. In *Logic in computer science*, pages 269–278, 2001.
3. E. Asarin and O. Maler. Achilles and the tortoise climbing up the arithmetical hierarchy. *Journal of Computer and System Sciences*, 57(3):389–398, dec 1998.
4. L. Blum, F. Cucker, M. Shub, and S. Smale. *Complexity and Real Computation.* Springer-Verlag, 1998.
5. L. Blum, M. Shub, and S. Smale. On a theory of computation and complexity over the real numbers; *np* completeness, recursive functions and universal machines. *Bulletin of the American Mathematical Society*, 21(1):1–46, July 1989.
6. O. Bournez. Achilles and the Tortoise climbing up the hyper-arithmetical hierarchy. *Theoretical Computer Science*, 210(1):21–71, 6 1999.
7. O. Bournez. *Complexité algorithmique des systèmes dynamiques continus et hybrides.* PhD thesis, École Normale Supérieure de Lyon, janvier 1999.
8. M. Bowles. United States technological enthusiasm and the british technological skepticism in the age of the analog brain. In *IEEE Annals of the History of Computing*, volume 4, pages 5–15, 1996.
9. V. Brattka. Computability over topological structures. In S. B. Cooper and S. S. Goncharov, editors, *Computability and Models*, pages 93–136. Kluwer Academic Publishers, New York, 2003.
10. M. Campagnolo, C. Moore, and J. F. Costa. An analog characterization of the Grzegorczyk hierarchy. *Journal of Complexity*, 18(4):977–1000, 2002.
11. M. L. Campagnolo. *Computational complexity of real valued recursive functions and analog circuits.* PhD thesis, Universidade Técnica de Lisboa, 2001.
12. J.-P. Demailly. *Analyse Numérique et Equations Différentielles.* Presses Universitaires de Grenoble, 1991.
13. G. Etesi and I. Németi. Non-Turing computations via Malament-Hogarth spacetimes. *International Journal Theoretical Physics*, 41:341–370, 2002.
14. D. Graça and J. F. Costa. Analog computers and recursive functions over the reals. *Journal of Complexity*, 19:644–664, 2003.
15. A. Grzegorczyk. Computable functionals. *Fundamenta Mathematicae*, 42:168–202, 1955.
16. T. Henzinger and J.-F. Raskin. Robust undecidability of timed and hybrid systems. *Hybrid systems: computation and control; second international workshop, hscc'99, berg en dal, the netherlands, march 29–31, 1999; proceedings*, 1569, 1999.
17. M. L. Hogarth. Does general relativity allow an observer to view an eternity in a finite time? *Foundations of physics letters*, 5:173–181, 1992.
18. D. Lacombe. Extension de la notion de fonction récursive aux fonctions d'une ou plusieurs variables réelles III. *Comptes rendus de l'Académie des Sciences Paris*, 241:151–153, 1955.
19. L. Lipshitz and L. A. rubel. A differentially algebraic replacement theorem, and analog computability. *Proceedings of the American Mathematical Society*, 99(2):367–372, February 1987.
20. C. Moore. Recursion theory on the reals and continuous-time computation. *Theoretical Computer Science*, 162(1):23–44, 5 1996.
21. J. Mycka. Infinite limits and R-recursive functions. *Acta Cybernetica*, 16:83–91, 2003.

22. J. Mycka. μ-recursion and infinite limits. *Theoretical Computer Science*, 302:123–133, 2003.
23. P. Odifreddi. *Classical recursion theory II.* North-Holland, 1999.
24. T. Ord. Hypercomputation: computing more than the Turing machine. Technical report, University of Melbourne, september 2002. available at http://www.arxiv.org/abs/math.lo/0209332.
25. P. Orponen. *Algorithms, languages and complexity*, chapter A survey of continuous-time computational theory, pages 209–224. Kluwer Academic Publishers, 1997.
26. B. Poizat. *Les petits cailloux.* aléas, 1995.
27. M. B. Pour-El. Abstract computability and its relation to the general purpose analog computer (some connections between logic, differential equations and analog computers). *Transactions of the American Mathematical Society*, 199:1–28, 1974.
28. E. Ramis, C. Deschamp, and J. Odoux. *Cours de mathématiques spéciales, tome 3, topologie et éléments d'analyse.* Masson, feb 1995.
29. H. Rose. *Subrecursion: Functions and Hierarchies.* Clarendon Press, 1984.
30. C. E. Shannon. Mathematical theory of the differential analyser. *Journal of Mathematics and Physics MIT*, 20:337–354, 1941.
31. H. Siegelmann. *Neural networks and analog computation - beyond the Turing limit.* Birkauser, 1998.
32. W. Thomson. On an instrument for calculating the integral of the product of two given functions. In *Proceedings of the royal society of London*, number 24, pages 266–276, 1876.
33. A. Turing. On computable numbers, with an application to the "Entscheidungsproblem". In *Proceedings of the london mathematical society*, volume 2, pages 230–265, 1936.
34. K. Weihrauch. *Computable Analysis.* Springer, 2000.
35. Q. Zhou. Subclasses of computable real valued functions. *Lecture Notes in Computer Science*, 1276:156–165, 1997.